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Group Contraction in Quantum Field Theory

Giuseppe Vitiello

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Abstract Group contraction plays a relevant rôle in spontaneously broken symmetry theories. Its physical meaning in connection with Bose condensation and the origin of macroscopic quantum systems is discussed.

Keywords Group contraction · Quantum field theory · Spontaneous breakdown of symmetry · Bose condensation

1 Introduction

The ordered patterns we observe in condensed matter physics and in high energy physics are created by the quantum dynamics. Many macroscopic systems exhibiting some kind of ordering, such as crystals, ferromagnets, superconductors, are described by the underlying quantum dynamics. Even the arrangement of some large scale structures in the Universe, as well as the ordering in biological systems appear to be the manifestation of the microscopic dynamics ruling the elementary components of these systems. My aim in this paper is to review, indeed, how the generation of ordered structures is explained in Quantum Field Theory (QFT) [18, 59, 61, 62, 65]: as we will see, the observed ordered patterns are generated by the *dynamical rearrangement of the symmetry* of the underlying dynamics. The mechanism which is at work, according to well established results of QFT, goes under the general name of *spontaneous breakdown of symmetry* and involves the physical phenomena of the *Bose condensation* and the mathematical structure of the (Ïnonü–Wigner) *group contraction* [31, 51, 52, 70, 71].

1.1 The Quantum Field Theory Framework

The systems studied by QFT are systems with infinitely many degrees of freedom and are described in terms of *operator fields*. One has the in-field and the out-field state space. Since,

G. Vitiello (✉)
Dipartimento di Fisica "E.R.Caianiello", Università di Salerno, and INFN Gruppo Collegato di Salerno, 84100 Salerno, Italy
e-mail: vitiello@sa.infn.it

at the quantum level, observations performed in the interaction region may drastically interfere with the interacting objects thus changing their nature, the only regions accessible to observations are those where the interaction forces can be safely assumed to not operate, far away (in space and in time) from the interaction region, i.e. the asymptotic regions (the in- and out-regions). Besides the asymptotic fields, one then also introduces dynamical or Heisenberg fields. These are the interacting fields in terms of which the dynamics is given. Since the interaction region is precluded from observation, Heisenberg fields are not observable. Observables are thus solely described in terms of asymptotic fields (these are called the quasiparticle fields in many body physics).

Such a situation is described by the Lhemann–Symanzik–Zimmermann (LSZ) formalism of QFT [10, 32, 59, 60], where the dynamics (the Lagrangian) is described in terms of the Heisenberg fields. The dynamical map expressing the Heisenberg fields in terms of the physical fields is known as the Haag expansion and is a weak relation in the sense that it only holds between expectation values over the physical state space. The set of the physical fields is supposed to be an irreducible set of fields and it may include bound states fields. Therefore, in general there is no one-to-one correspondence between Heisenberg fields and physical fields.

It is important to mention that in some cases the interaction cannot be switched off. Thus free, i.e. non-interacting, fields cannot be always identified. In this paper, however, I will consider those cases where one can always safely assume that interaction can be considered switched off in some asymptotic regions.

Interacting field equations, derived from the Lagrangian describing the evolution of the system in space-time, are thus the motion equations for the Heisenberg fields. The dynamical equations, however, *do not* define completely the dynamical problem: one must *also* specify the space of the states on which the dynamics has to be realized. Only in that case the dynamical problem is well defined: indeed, operator fields are mathematically meaningful objects only when the state space on which they operate is given. This has a deep physical consequence: it means that the possibility exists that the *same* dynamical equations for the *same* set of Heisenberg fields may be realized in different, i.e. *physically inequivalent*, spaces of states and therefore, depending on the state space one works with, they may produce *different* dynamical outputs.

The existence of physically inequivalent state spaces, i.e. of infinitely many unitarily inequivalent representations of the canonical (anti-)commutation relations, is allowed in QFT since there the systems have infinitely many degrees of freedom. In quantum mechanics, on the contrary, since the number of degrees of freedom is finite, all the representations of the canonical (anti-)commutation relations are unitarily, and thus physically, equivalent, as required by the von Neumann theorem [10, 67]. Thus, in QFT, provided mathematical care is adopted, transitions among physically inequivalent representations describe the system phase transitions (such as transition from the normal phase to the superconducting phase of a metal).

1.2 Spontaneous Breakdown of Symmetry

The symmetry properties of the Heisenberg field equations, i.e. the symmetry of the dynamics (in this paper I only consider continuous symmetries), may appear "broken" or rearranged into different symmetry patterns at level of the physical asymptotic fields. For example, in the case of the ferromagnets the Lagrangian from which the Heisenberg equations are derived is invariant under the spin rotational $SU(2)$ transformations of the Heisenberg field operators. On the other hand, the observable system, which is described in terms of quasiparticle (or physical) field operators, is characterized by its non-zero magnetization. The appearance of the privileged direction into which the magnetization points and its non vanishing value signal that the original $SU(2)$ isotropy at the level of the Heisenberg fields has been broken at the phenomenological level and the ferromagnetic ordering has been realized. Order thus appears as a manifestation of the symmetry breakdown, or, in other words, as lack of symmetry. The study of the equations for the quasiparticle fields shows indeed that they are not invariant under the $SU(2)$ group. Thus there are two sets of equations with different invariance properties: on one side, the dynamical equations derived from the Lagrangian, which remain invariant when the Heisenberg field operators undergo the $SU(2)$ transformations; on the other side, the equations for the quasiparticle field operators, in terms of which observables are expressed, which are not invariant under the $SU(2)$ transformations.

In full generality, I denote by G the transformation group under which the basic Lagrangian is invariant. The general questions concerning which one is the transformation group G' under which the equations for quasiparticle or physical fields are invariant, how the “symmetry rearrangement” $G \to G'$ occurs and which one is its physical meaning are the ones addressed in this paper.

I will not consider the case of “explicit” breakdown of symmetry. This is the case where one adds to the Lagrangian, assumed to be invariant under the continuous group G, a “symmetry breaking” term. Instead, I will discuss the “spontaneous” breakdown of symmetry. It occurs when the ground state or vacuum state $|0\rangle$ is not invariant under the continuous group G under which the Lagrangian is invariant. In other words, when at least some of the generators of G do not annihilate the ground state $|0\rangle$. The vacuum which is invariant under the symmetry group (and thus no breakdown of symmetry occurs) is called the “normal” or symmetric vacuum.

I observe that, under convenient boundary conditions, in principle every one among the possible non-symmetric vacua can be realized in Nature. In the ferromagnets, for example, the magnetization may point in any possible direction and the magnetization strength may in principle assume any value up to a saturation limit: the system, driven by its dynamics, “spontaneously” sets to the state characterized by a specific magnetization under given boundary conditions. This is why the breakdown of the symmetry is said to be “spontaneous”. As a consequence of this, the original invariance of the Lagrangian under the group G may manifest itself into many “different” symmetry patterns at the physical level. Since the magnetization fully characterizes the non-symmetric vacuum of a given representation, it acts as a label for the inequivalent representations. Each one of these representations describes a different physical phase of the system. Observables, such as the magnetization, good for labeling different symmetry patterns (or physical phases) are called order parameters. In this way it happens that the same basic dynamics (same Lagrangian with given invariance group G) may manifest itself into a variety of stable symmetry patterns at the level of the observables, each corresponding to different boundary conditions or to different ranges of the values of the theory parameters, and specified by the order parameter value. Changes occurring in the order parameter describe transitions among the system physical phases (phase transitions).

Since physical theories need to be tested by observations, the spaces which will be considered in the following will be the ones where the physical (in- or out-) field operators are realized. This introduces crucial constraints in the mathematical derivations, which account for important features in the way the original invariance of the Lagrangian manifests itself in the observable symmetry patterns.

I will not consider the rôle of the temperature usually contrasting the emergence of ordered patterns, or inducing symmetry restoration. I will also omit to consider questions related with renormalization problems.

The terminology "breakdown of symmetry" might suggest that the invariance of the Lagrangian under the continuous symmetry group G is in some way lost when symmetry is broken. However, the invariance of the Lagrangian means that the generators of the group G commute with the Hamiltonian and this determines the constants of motion. Therefore, for internal consistency, the invariance cannot simply disappear. On the other hand, the question arises of which one is the relation between the symmetry group for the Heisenberg field equations and the one for the physical field equations. As previously mentioned, the mapping between the Heisenberg fields and the physical fields is displayed through the dynamical map. Due to the nonlinear character of the dynamical map, which reflects nonlinear dynamical effects, one naturally expects that the symmetry properties at the level of the Heisenberg field operators may manifest themselves through a mechanism of "dynamical rearrangement" $G \to G'$ at the level of physical fields. Under quite general conditions, it turns out that G' is the group contraction of G.

One reason why the symmetry group G' may be different from G is based on the fact that any observation on a system described by fields is a collection of local observations. Therefore, there always exists the possibility that in each local observation one misses an infinitesimal contribution of the order of magnitude of $\frac{1}{V}$, with the volume $V \to \infty$. This missing effect can be accumulated as a finite amount when it is integrated over the whole system, thus producing the difference $G' \neq G$: it is responsible of the group contraction phenomenon. Such a local infinitesimal contribution is called infrared effect [39, 53].

In the following I will proceed by discussing some physically relevant models. However, the conclusions can be extended in a quite general way to other QFT models. Indeed, group contraction has been shown to occur in models invariant under $SU(n)$, $SO(n)$, chiral $SU(2) \times SU(2)$, $SU(3) \times SU(3)$, etc. (see [18] and [59] and references therein quoted). One might classify the different cases into three categories R_i, $i = 1, 2, 3$ [18].

In the case R_1 the dynamical groups are Abelian; the dynamical and phenomenological symmetry have the same algebraic structure: the rearrangement leads to a trivial contraction of the basic symmetry algebra; Heisenberg and asymptotic fields provide different realizations of this algebra. Examples include spontaneous breakdown of phase, chiral phase and scale invariance [38, 44, 58].

In the case R_2 the dynamical rearrangement manifestly leads to a contraction of the basic symmetry algebra. Examples are given in Refs. [33, 39, 43, 53, 54, 58, 62–64] and include, e.g., the case of a scalar isotriplet, the ferromagnet discussed below, the chiral $SU(2) \times SU(2)$ symmetry [33] realized by nonlinear transformation of the pion field, the $SU(3)$ group in a linear approximation of solid state systems as $T - t$ Jahn–Teller systems [63, 69].

In the case R_3 the generators of the phenomenological symmetry do not form a closed algebra. By enlarging the set of generators one can complete the algebra. In Ref. [44] such an extreme case of rearrangement is studied for an $SU(2)$ invariant model.

The dynamical rearrangement of continuous spatial translation group into discrete translation group leading to lattice structure has been also studied with particular reference to crystal formation [59, 60]. In that case the Nambu–Goldstone bosons are the quanta of the elastic waves, i.e. the phonons.

The paper is organized as follows: the spontaneous breakdown of the $SU(2)$ and of the $U(1)$ symmetry will be studied in Sects. 2 and 3, respectively. The dynamical rearrangement of the symmetry and the infrared effect will be discussed in Sects. 4 and 5, respectively. Section 6 is devoted to conclusions and further remarks.

2 Spontaneous Breakdown of $SU(2)$ Symmetry

A physically interesting example is the itinerant electron model of the ferromagnet. The localized spin model can be studied in a similar way, see [54]. Another interesting $SU(2)$ example is the one of the isospin vector fields reported in [39]. Let $\psi(x)$ denote the electron field:

$$\psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}, \tag{1}$$

with $\uparrow$ and $\downarrow$ denoting the field spin up or down, respectively. Under $SU(2)$ $\psi(x)$ transforms as

$$\psi(x) \to \psi'(x) = \exp(i\theta_i\lambda_i)\psi(x), \quad i = 1, 2, 3, \tag{2}$$

with $\lambda_i = \frac{\sigma_i}{2}$, σ_i the Pauli matrices, and θ_i a triplet of real continuous group parameters.

One does not need to specify the explicit form of the Lagrangian. It is only required to be invariant under the $SU(2)$ group of rotations (2). Let $S^{(i)}(x),\ i = 1, 2, 3$, be the $SU(2)$ generators:

$$[S^{(i)}(x), S^{(j)}(x)] = i\epsilon_{ijk}S^{(k)}(x). \tag{3}$$

The explicit form of the generators $S^{(i)}(x)$ in terms of the anticommuting fields $\psi(x)$ can be given for example by $S_\psi^{(i)}(x) = \psi^\dagger(x)\frac{\sigma_i}{2}\psi(x)$. Most of the conclusions will be however independent of the specific form of $S^{(i)}(x)$. In the case of localized spins, one may introduce $S^{(i)}(x_l)$ and the (total) $SU(2)$ generators

$$S^{(i)} = \sum_l S^{(i)}(x_l), \quad i = 1, 2, 3, \tag{4}$$

$$[S^{(i)}, S^{(j)}] = i\epsilon_{ijk}S^{(k)}. \tag{5}$$

The invariance of the Lagrangian under $SU(2)$ implies: $\mathcal{L}[\psi(x)] = \mathcal{L}[\psi'(x)]$; the ground state $|0\rangle$ is however assumed to be not invariant under the full $SU(2)$ group but only under the subgroup $U(1)$ of the rotations around the 3rd axis in the spin-space.

The Green's function generating functional is

$$\begin{aligned} W[J, j, n] = \frac{1}{N}\int [d\psi][d\psi^\dagger]\exp i\int dt\{\mathcal{L}[\psi(x)] + J(x)\psi(x) + \psi^\dagger(x)J(x) \\ + j^\dagger(x)S_\psi^{(-)}(x) + S_\psi^{(+)}(x)j(x) + S_\psi^{(3)}(x)n(x) - i\epsilon S_\psi^{(3)}(x)\}, \end{aligned} \tag{6}$$

where N is the normalization factor

$$N[J, j, n] = \int [d\psi][d\psi^\dagger]\exp i\int dt\{\mathcal{L}[\psi(x)] - i\epsilon S_\psi^{(3)}(x)\}. \tag{7}$$

$S_\psi^{(\alpha)}(x)$, $\alpha = \pm, 3$, $S_\psi^{(\pm)}(x) \equiv S_\psi^{(1)}(x) \pm iS_\psi^{(2)}(x)$, is the spin density made of $\psi(x)$. The electron fields $\psi(x)$, $\psi(x)^\dagger$ and their sources J, $J^\dagger$ anticommute; the sources j are commuting c-numbers. The ϵ-term has been introduced in order to include in the functional integral the information of the spontaneous breakdown of the symmetry [38, 60] and the limit $\epsilon \to 0$ has to be taken at the end of the computation. In the functional integral formalism the functional average $\langle F[\psi]\rangle$ agrees with the ground state expectation value of the $T(F[\psi])$ where T denotes the chronological products of the Heisenberg fields $\psi(x)$ and $\psi^\dagger(x)$.

$$\langle F[\psi]\rangle = \langle 0|T(F[\psi])|0\rangle. \tag{8}$$

The ground state expectation values of chronological products of $\psi(x)$ and $\psi^\dagger(x)$, i.e. the Green's functions, are now obtained by repeated functional derivatives of $W[J, j, n]$ with respect to the relative sources $\frac{\partial}{\partial J^\dagger}$ and $\frac{\partial}{\partial J}$ followed by the limits of J, j and n going to zero. The presence of the source terms with j and n allows the study of the behavior of the spin densities without specifying the dependence of $S_\psi^{(i)}$ on ψ.

For θ_i infinitesimal $S_\psi^{(i)}(x)$ transforms as

$$S_\psi^{(i)}(x) \to S_\psi^{(i)'}(x) = S_\psi^{(i)}(x) - \theta_j \epsilon_{ijk} S_\psi^{(k)}(x). \tag{9}$$

Now, putting $J = 0 = n$ and performing the change of variables (2) in the numerator of (6), one gets

$$\frac{\partial W}{\partial \theta_l} = 0. \tag{10}$$

By operating with $\frac{\delta}{\delta j(y)}$ on this and putting $j = 0$ one then obtains

$$(\epsilon_{1lk} + i\epsilon_{2lk})\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x \langle S_\psi^{(k)}(x) S_\psi^{(+)}(y)\rangle_\epsilon. \tag{11}$$

Similarly, operating with $\frac{\delta}{\delta j^\dagger(y)}$ and putting $j = 0$ leads to

$$(\epsilon_{1lk} - i\epsilon_{2lk})\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x \langle S_\psi^{(k)}(x) S_\psi^{(-)}(y)\rangle_\epsilon. \tag{12}$$

These two last equations lead to

$$\epsilon_{1lk}\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x \langle S_\psi^{(k)}(x) S_\psi^{(1)}(y)\rangle_\epsilon. \tag{13}$$

$$\epsilon_{2lk}\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x \langle S_\psi^{(k)}(x) S_\psi^{(2)}(y)\rangle_\epsilon. \tag{14}$$

From these equations, for $l = 1$ and $l = 2$, it follows:

$$\epsilon\int d^4x \langle S_\psi^{(2)}(x) S_\psi^{(1)}(y)\rangle_\epsilon = 0, \tag{15}$$

$$\langle S_\psi^{(3)}(y)\rangle_\epsilon = \epsilon\int d^4x \langle S_\psi^{(1)}(x) S_\psi^{(1)}(y)\rangle_\epsilon, \qquad \langle S_\psi^{(3)}(y)\rangle_\epsilon = \epsilon\int d^4x \langle S_\psi^{(2)}(x) S_\psi^{(2)}(y)\rangle_\epsilon, \tag{16}$$

and for and $l = 3$:

$$\langle S_\psi^{(1)}(y)\rangle_\epsilon = \langle S_\psi^{(2)}(y)\rangle_\epsilon = 0. \tag{17}$$

Write now

$$\langle S_\psi^{(i)}(x)S_\psi^{(i)}(y)\rangle_\epsilon = i\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\rho_{(i)}(p)\left(\frac{1}{p_0-\omega_p+i\epsilon a_i}-\frac{1}{p_0+\omega_p p-i\epsilon a_i}\right) + \text{c.c.}, \quad i=1,2,3. \tag{18}$$

Here the notation is $p(x-y) = -\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) + ip_0(t_x-t_y)$ and ω_p is the energy of a quasiparticle which is a bound state of electrons. It existence is proved by showing that the spectral density $\rho_i(p)$ is not zero. The explicit dynamical calculation can be done provided the specific form of the Lagrangian is assigned (see e.g. [54] and [60]). It is however remarkable that the general treatment based on symmetry considerations proves the existence of such a bound state in full generality (i.e. in a model independent way since the Lagrangian form has not been specified except for its invariance properties). The continuum contribution (c.c.) in (18) comes from states which contain more than one quasiparticle. The singularities in the Feynman Green's functions are defined as usual by $\omega_p - i\eta$ with infinitesimal η. In (18) $a_i = \frac{\eta}{\epsilon}$. Note that since $S_\psi^{(i)}$ are Hermitian $\rho_i(p)$ cannot be negative.

By operating with $[\frac{\delta}{\delta j^\dagger(z)}][\frac{\delta}{\delta j(y)}]$ and $[\frac{\delta}{\delta j(z)}][\frac{\delta}{\delta j^\dagger(y)}]$ on (10), putting then $j=0$ and subtracting leads to

$$\langle S_\psi^{(1)}(x)S_\psi^{(1)}(y)\rangle_\epsilon = \langle S_\psi^{(2)}(x)S_\psi^{(2)}(y)\rangle_\epsilon, \tag{19}$$

which gives $\rho_1(p)=\rho_2(p)$ and $a_1=a_2$. The magnetization is given by $g\mu_B\langle S_\psi^{(3)}(x)\rangle_\epsilon$ with μ_B the Bohr magneton. The notation $M(\epsilon) = \langle S_\psi^{(3)}(x)\rangle_\epsilon$ will be used, and

$$M = \lim_{\epsilon\to 0} M(\epsilon) \tag{20}$$

(16) and (18) say that in order to have a non-zero M, there should be a bound state of gapless energy $\omega_p = 0$ at $p=0$ (the Goldstone theorem). Indeed they give

$$M(\epsilon) = i\epsilon\Delta_i(\epsilon,0), \quad i=1,2, \tag{21}$$

which can lead to non-vanishing M with $\epsilon\to 0$ only when $\omega_p=0$ at $p=0$. One further has

$$M = \frac{2\rho}{a}, \quad i=1,2. \tag{22}$$

In (22) $\rho=\rho_1=\rho_2$ and $a=a_1=a_2$ and in (21) it has been used

$$\Delta_i(\epsilon,p) = \rho_{(i)}(p)\left(\frac{1}{p_0-\omega_p+i\epsilon a_i}-\frac{1}{p_0+\omega_p-i\epsilon a_i}\right). \tag{23}$$

In the case of localized spins, the integration of $\mathbf{p}$ is confined to the domain $-\frac{\pi}{d} < p_i < \frac{\pi}{d}$, where d is the lattice length and derivation of (21) requires use of the formula

$$\frac{v}{(2\pi)^3}\sum_l e^{-ipx_l} = \delta^{(3)}(p), \tag{24}$$

with v the volume of unit lattice.

Now I calculate ρ. The total spin in the third direction is NM, where N is the number of lattice points. Then the ground state expectation value of $\mathbf{S}^2$ is given by

$$\langle 0|\mathbf{S}^2|0\rangle = NM(NM+1). \tag{25}$$

By assuming $t_k < t_l$ in (18), with $i = 1, 2$, and performing the limit $t_k \to t_l$ (same result is obtained by assuming $t_l < t_k$), one finds, using (24), that

$$\langle 0|S^i S^i|0\rangle = \rho N \quad \text{for } i = 1, 2. \tag{26}$$

Therefore, $\langle 0|\mathbf{S}^2|0\rangle = 2\rho N + (NM)^2$ and, comparing with (25), $\rho = \frac{1}{2}M$, which gives $a = 1$ (cf. (22)).

In conclusion, (20) along with non-zero M requires the existence of gapless bosons, i.e. the magnons, which are the Nambu–Goldstone (NG) bosons of the breakdown of the spin $SU(2)$ symmetry of the Lagrangian. In practical computations the magnon is a bound state of electrons and is treated by the Bethe–Salpeter equation [54, 60]. As well known in the theory of the ferromagnets, the magnons are the long range correlations responsible for ferromagnetic ordering [27, 45, 60, 68]. They are the spin wave quanta. Thus, ordering is originated from the spontaneous breaking of the $SU(2)$ symmetry, through the dynamical generation of the NG gapless bound states (the magnons).

In the proof of the Goldstone theorem presented above the system volume is considered to be infinite. This is a reasonable working assumption since observations are always local and therefore the system volume V may be taken to be infinite. This is also the case of the so called thermodynamic limit where the limit to the infinite number of degrees of freedom and to the infinite volume is taken is a way that the density remains finite. The spatial integration domain, e.g. in (18), thus extends to infinity and this is crucial in picking up the zero-momentum contribution in the two-point Green's function. As discussed below, the dynamical rearrangement of the symmetry occurs since terms of the order of $\frac{1}{V}$ are missing in local observations.

It is, however, interesting to consider the boundaries effects on the dynamics, due to the finiteness of the system volume. For example, in some cases it is necessary to consider how the ordering induced by the NG condensation gets distorted in the vicinity of the system boundaries and how "defects" (non-homogeneous condensation) appear. One can then show that the NG particle acquires an effective non-zero mass due to finite volume effects [4, 5, 65]. The effective mass of the NG mode reflects on the correlation length and thus it is directly related to the size of the ordered domain. Such volume effects can be also related with temperature effects. For sake of shortness, here I do not discuss further these topics, see [4, 5, 21, 65].

3 The Anderson–Higgs–Kibble Mechanism

I consider now the example of the complex scalar field with $U(1)$ local gauge symmetry. Let $\phi(x)$ denote the Heisenberg complex scalar field interacting with the Heisenberg gauge field $A_\mu(x)$ [7, 28, 34, 40]. The Lagrangian density $\mathcal{L}[\phi(x), \phi^*(x), A_\mu(x)]$ is invariant under the global and the local gauge transformations:

$$\phi(x) \to e^{i\theta}\phi(x), \qquad A_\mu(x) \to A_\mu(x), \tag{27}$$

$$\phi(x) \to e^{ie_0\lambda(x)}\phi(x), \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x), \tag{28}$$

respectively, where $\lambda(x) \to 0$ for $|x_0| \to \infty$ and/or $|\mathbf{x}| \to \infty$. I will use the Lorentz gauge:

$$\partial_\mu A^\mu(x) = 0, \tag{29}$$

and put

$$\phi(x) = \frac{1}{\sqrt{2}}[\psi(x) + i\chi(x)], \qquad \rho(x) = \psi(x) - \langle\psi(x)\rangle_\epsilon. \tag{30}$$

Spontaneous breakdown of symmetry is introduced through the condition

$$\langle 0|\phi(x)|0\rangle \equiv \tilde{v} \neq 0, \tag{31}$$

with $\tilde{v}$ constant. The generating functional, including the gauge constraint through a functional delta-like term, is [40]

$$\begin{aligned} W[J,K] = \frac{1}{N}\int [dA_\mu][d\phi][d\phi^*][dB] \exp i \int d^4x \{\mathcal{L}(x) + B(x)\partial^\mu A_\mu(x) \\ + K^*\phi + K\phi^* + J^\mu(x)A_\mu(x) + i\epsilon|\phi(x) - v|^2\}, \\ N = \int [dA_\mu][d\phi][d\phi^*][dB] \exp i \int d^4x \{\mathcal{L}(x) + i\epsilon|\phi(x) - v|^2\}. \end{aligned} \tag{32}$$

$B(x)$ is an auxiliary field which guarantees the gauge condition. As in the case studied in the previous section, the Ward–Takahashi identities are obtained. In particular the following pole structure is obtained for the two-point functions [40]:

$$\langle\chi(x)\chi(y)\rangle = \lim_{\epsilon\to 0}\left\{\frac{i}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}\frac{Z_\chi}{p^2 + i\epsilon a_\chi} + (\text{contin contr})\right\}, \tag{33}$$

$$\langle B(x)\chi(y)\rangle = \lim_{\epsilon\to 0}\left\{\frac{-i}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}\frac{e_0\tilde{v}}{p^2 + i\epsilon a_\chi}\right\}, \tag{34}$$

$$\langle B(x)A^\mu(y)\rangle = \partial_x^\mu \frac{i}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}\frac{1}{p^2}, \tag{35}$$

$$\langle B(x)B(y)\rangle = \lim_{\epsilon\to 0}\frac{-i}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}\frac{(e_0\tilde{v})^2}{Z_\chi}\left[\frac{1}{p^2 + i\epsilon a_\chi} - \frac{1}{p^2}\right]. \tag{36}$$

Equation (33) shows, in a way similar to the cases considered in the previous section, that the field χ is the NG massless mode. Equation (36) shows that the model contains in the present case also a massless negative norm state (ghost). The absence of cut singularities in last three of these propagators suggests that $B(x)$ obeys a free field equation. Moreover, it can be shown that a massive vector field U^μ also exists in the theory and that the NG and the ghost modes do not appear in the physical particle spectrum (the Anderson–Higgs–Kibble mechanism) [40]. Using $B(x) \to B(x) + \lambda(x)$ in (32) gives, after functional derivative,

$$\langle\partial^\mu A_\mu(x)\rangle_{\epsilon,J,K} = 0. \tag{37}$$

The two-point functions shown above provide the tools necessary to derive the dynamical maps. The result is the following [40]:

$$S = :S[\rho_{in}, U^\mu_{in}, \partial(\chi_{in} - b_{in})]:, \tag{38}$$

$$\phi(x) = :\exp\left\{ i \frac{Z_\chi^{\frac{1}{2}}}{\tilde{v}} \chi_{in}(x) \right\} [\tilde{v} + Z_\rho^{\frac{1}{2}} \rho_{in}(x) + F[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})]]:, \tag{39}$$

$$A^\mu(x) = Z_3^{\frac{1}{2}} U_{in}^\mu(x) + \frac{Z_\chi^{\frac{1}{2}}}{e_0 \tilde{v}} \partial^\mu b_{in}(x) + :F^\mu[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})]:, \tag{40}$$

where the functionals F and F^μ are to be determined within a particular model. It will be also used the notation $A^{0\mu}(x) \equiv A^\mu(x) - e_0\tilde{v}{:}\partial^\mu b_{in}(x){:}$. In (38–40) χ_{in} denotes the NG mode, b_{in} the ghost mode, U_{in}^μ the massive vector field and ρ_{in} the massive matter field. Their field equations are

$$\partial^2 \chi_{in}(x) = 0, \qquad \partial^2 b_{in}(x) = 0, \qquad (\partial^2 + m_\rho^2)\rho_{in}(x) = 0, \tag{41}$$

$$(\partial^2 + m_V^2) U_{in}^\mu(x) = 0, \qquad \partial_\mu U_{in}^\mu(x) = 0. \tag{42}$$

with $m_V^2 = Z_3 Z_\chi^{-1}(e_0\tilde{v})^2$. One also has

$$B(x) = e_0 \tilde{v} Z_\chi^{-\frac{1}{2}} [b_{in}(x) - \chi_{in}(x)]. \tag{43}$$

The field equations for B and A_μ are

$$\partial^2 B(x) = 0, \qquad -\partial^2 A_\mu(x) = j_\mu(x) - \partial_\mu B(x), \tag{44}$$

with $j_\mu(x) = \delta\mathcal{L}(x)/\delta A^\mu(x)$. Requiring that the current j_μ is the only source of the gauge field A_μ in any observable process amounts to impose the condition: ${}_p\langle b|\partial_\mu B(x)|a\rangle_p = 0$, i.e. from (44)

$$(-\partial^2)_p\langle b|A^0{}_\mu(x)|a\rangle_p =_p \langle b|j_\mu(x)|a\rangle_p, \tag{45}$$

where $|a\rangle_p$ and $|b\rangle_p$ denote two generic physical states. Equation (45) *are the classical Maxwell equations*. The condition ${}_p\langle b|\partial_\mu B(x)|a\rangle_p = 0$ leads to the Gupta–Bleuler-like condition

$$[\chi_{in}^{(-)}(x) - b_{in}^{(-)}(x)]|a\rangle_p = 0, \tag{46}$$

where $\chi_{in}^{(-)}$ and $b_{in}^{(-)}$ are the positive-frequency parts of the corresponding fields. Thus, χ_{in} and b_{in} do not participate to any observable reaction. Note in fact that they are present in the S matrix in the combination $(\chi_{in} - b_{in})$ (cf. (38)). This fact along with the appearance in theory of the massive vector field $U_\mu(x)$ is referred to as the Anderson–Higgs–Kibble mechanism [7, 8, 23, 28, 34]. I stress that, although the NG particles χ_{in} (and the ghost field b_{in}) do not show up in the particle spectrum, nevertheless their condensation characterizes the physical state structure (see (46)) and their rôle is crucial in recovering the internal consistency of the invariance properties of the theory. It is also remarkable that the NG boson condensation leads to the classic Maxwell equations, thus exhibiting macroscopic manifestations of the microscopic dynamics. Moreover, one can also see that, when the vacuum is not translationally invariant, the NG fields produce observable effects in the formation of macroscopically behaving extended objects, such as kinks, vortices, etc. [40, 41, 59, 60, 65].

4 Dynamical Rearrangement Symmetry and Group Contraction

In the present section I want to study how the invariance of theory manifests itself at the level of the physical (quasiparticle) fields when the G symmetry is spontaneously broken. The problem to solve is therefore the one of finding the symmetry group G' under which the free field equations are invariant. In general G' turns out to be the group contraction of G. This means that G' contains a subgroup of transformations which induce translations of the NG boson fields and thus it describes the condensation of the NG modes in the ground state. In this way ordered patterns are generated. These ordered patterns constitute the macroscopic manifestation of the symmetry breakdown. The NG modes are long range correlation modes and leads us to recognize the collective nature of the ordering occurring as a manifestation of the spontaneous breakdown of the symmetry. In this connection, I recall that in the Andreson–Higgs–Kibble mechanism discussed above the classical Maxwell equations have already been obtained as macroscopic manifestations of the microscopic dynamics. In this section it will be discussed the occurrence of macroscopic (classically behaving) quantities and how the microscopic dynamics manifests into *macroscopic quantum systems*.

4.1 The $SU(2)$ Ferromagnetic Model

The dynamical map of the Heisenberg electron fields $\psi(x)$ in terms of the quasielectron field $\phi(x)$ and of the magnon field $B(x)$ is:

$$\psi(x) = \Psi(\phi(x), B(x)). \tag{47}$$

The dynamical rearrangement consists in the change of the continuous symmetry group $G \equiv SU(2)$ into the transformation group G', under which the equations for the quasiparticle fields are invariant:

$$\psi'(x) = \Psi(g[\phi(x)]), \quad g \in G' \tag{48}$$

where $\psi'(x)$ is the transformed of $\psi(x)$ under $SU(2)$ according to (2).

The magnon field $B(x)$ is a gapless bound state of the electron field. In the present discussion I omit considering other fields such as the e.m. field. We want to know which one is the group G'. The boson field for the magnons is introduced as

$$B(x) = \int \frac{d^3k}{(2\pi)^{3/2}} B_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t}, \qquad B^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}} B^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_k t}, \tag{49}$$

with commutation relations

$$[B(x), B^\dagger(y)]_{t_x - t_y} = \delta(\mathbf{x} - \mathbf{y}), \qquad [B(x), B^\dagger(y)] = [B(x), B^\dagger(y)] = 0. \tag{50}$$

The fields (49) satisfy the equations

$$K(\overrightarrow{\partial})B^\dagger(x) = 0, \qquad B(x)K(\overleftarrow{\partial}) = 0, \tag{51}$$

where

$$K(\overrightarrow{\partial}) = -\left(i\frac{\overrightarrow{\partial}}{\partial t} + \omega\right). \tag{52}$$

The free field equations for the quasielectron are

$$\Lambda(\overrightarrow{\partial})\phi(x) = 0, \qquad \phi^\dagger(x)\Lambda(\overleftarrow{\partial}) = 0, \tag{53}$$

where $\Lambda(\overrightarrow{\partial})$ denotes the partial derivative operator, including the quasielectron mass term, which is appropriate for the model specified by the Lagrangian one considers.

In the following it is convenient to work with the Heisenberg spin operator densities $S_\psi^{(i)}(x)$, $i = 1, 2, 3$, which transform under $SU(2)$ as in (9).

To obtain the expressions of the S-matrix $\mathcal{S}$ and of the spin densities $S^{(i)}(x)$ in terms of ϕ and of B fields, $\mathcal{S}(\phi, \phi^\dagger, B, B^\dagger)$ and $S^{(i)}(x, \phi, \phi^\dagger, B, B^\dagger)$, the functional formalism together with the LSZ formula is used. One has [53]

$$\mathcal{S}(\phi, \phi^\dagger, B, B^\dagger) = \langle :\exp[-iA(\phi, \phi^\dagger, B, B^\dagger)]:\rangle \tag{54}$$

and

$$S^{(i)}(\phi, \phi^\dagger, B, B^\dagger) = \langle S_\psi^{(i)}(x) :\exp[-iA(\phi, \phi^\dagger, B, B^\dagger)]:\rangle \tag{55}$$

where $i = 1, 2, 3$ and

$$\begin{aligned} A(\phi, \phi^\dagger, B, B^\dagger) = \int d^4x [&\rho^{-1/2} B(x) K(\overrightarrow{\partial}) S_\psi^{(-)}(x) + \rho^{-1/2} S_\psi^{(+)}(x) K(-\overleftarrow{\partial}) B^\dagger(x) \\ &+ Z^{-1/2} \phi^\dagger(x) \Lambda(-\overrightarrow{\partial}) \psi(x) + Z^{-1/2} \psi^\dagger(x) \Lambda(-\overleftarrow{\partial}) \phi(x)]. \end{aligned} \tag{56}$$

Here Z is the wave function renormalization of the electron and $\rho = \frac{1}{2}M$. As usual the symbol $:\cdots:$ denotes normal product ordering and $\langle\cdots\rangle$ denotes functional average. Our task is to find the transformations for $\phi, \phi^\dagger, B, B^\dagger$ in (54) and (55) which leave invariant their field equations and such that the transformation (9) of $S^{(i)}(\phi, \phi^\dagger, B, B^\dagger)$ is induced. I denote the transformed fields by $\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger$ and require they satisfy the equations for the quasiparticles

$$K(\overrightarrow{\partial}) B_\theta^\dagger(x) = 0, \qquad B_\theta(x) K(\overleftarrow{\partial}) = 0, \tag{57}$$

$$\Lambda(\overrightarrow{\partial}) \phi_\theta(x) = 0, \qquad \phi_\theta^\dagger(x) \Lambda(\overleftarrow{\partial}) = 0, \tag{58}$$

and that

$$\frac{\partial}{\partial \theta_l} \mathcal{S}(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger) = 0, \tag{59}$$

$$\frac{\partial}{\partial \theta_l} S^i(x, \phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger) = -\epsilon_{ilk} S^k(x, \phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger). \tag{60}$$

I use now the transformed fields in (56) and obtain equations for these fields implied by (59) and (60) by following the steps which can be found in Ref. [53] The equations eventually obtained are:

$$\begin{gathered} \frac{\partial}{\partial \theta_1} B_\theta(x) = i\left(\frac{M}{2}\right)^{1/2}, \qquad \frac{\partial}{\partial \theta_1} B_\theta^\dagger(x) = -i\left(\frac{M}{2}\right)^{1/2}, \\ \frac{\partial}{\partial \theta_1} \phi_\theta(x) = 0, \qquad \frac{\partial}{\partial \theta_1} \phi_\theta^\dagger(x) = 0, \end{gathered} \tag{61}$$

$$\begin{gathered} \frac{\partial}{\partial \theta_2} B_\theta(x) = -\left(\frac{M}{2}\right)^{1/2}, \qquad \frac{\partial}{\partial \theta_2} B_\theta^\dagger(x) = -\left(\frac{M}{2}\right)^{1/2}, \\ \frac{\partial}{\partial \theta_2} \phi_\theta(x) = 0, \qquad \frac{\partial}{\partial \theta_2} \phi_\theta^\dagger(x) = 0, \end{gathered} \tag{62}$$

$$\begin{aligned} &\frac{\partial}{\partial\theta_3}B_\theta(x) = -iB_\theta(x), \qquad \frac{\partial}{\partial\theta_3}B_\theta^\dagger(x) = iB_\theta^\dagger(x), \\ &\frac{\partial}{\partial\theta_3}\phi_\theta(x) = i\lambda_3\phi_\theta(x), \qquad \frac{\partial}{\partial\theta_2}\phi_\theta^\dagger(x) = -i\phi_\theta^\dagger(x)\lambda_3, \end{aligned} \tag{63}$$

and using the conditions $\phi_\theta(x) = \phi(x)$, $B_\theta(x) = B(x)$, etc., at $\theta = 0$, gives

$$\begin{aligned} &\phi(x) \to \phi_\theta(x) = \phi(x), \qquad \phi^\dagger(x) \to \phi_\theta^\dagger(x) = \phi^\dagger(x), \\ &B(x) \to B_\theta(x) = B(x) + i\theta_1\left(\frac{M}{2}\right)^{\frac{1}{2}}, \\ &B^\dagger(x) \to B_\theta^\dagger(x) = B^\dagger(x) - i\theta_1\left(\frac{M}{2}\right)^{\frac{1}{2}}, \end{aligned} \tag{64}$$

for $\theta_2 = \theta_3 = 0$,

$$\begin{aligned} &\phi(x) \to \phi_\theta(x) = \phi(x), \qquad \phi^\dagger(x) \to \phi_\theta^\dagger(x) = \phi^\dagger(x), \\ &B(x) \to B_\theta(x) = B(x) - \theta_2\left(\frac{M}{2}\right)^{\frac{1}{2}}, \\ &B^\dagger(x) \to B_\theta^\dagger(x) = B^\dagger(x) - \theta_2\left(\frac{M}{2}\right)^{\frac{1}{2}}, \end{aligned} \tag{65}$$

for $\theta_1 = \theta_3 = 0$, and

$$\begin{aligned} &\phi(x) \to \phi_\theta(x) = e^{i\theta_3\lambda_3}\phi(x), \qquad \phi^\dagger(x) \to \phi_\theta^\dagger(x) = \phi^\dagger(x)e^{-i\theta_3\lambda_3}, \\ &B(x) \to B_\theta(x) = e^{-i\theta_3}B(x), \qquad B^\dagger(x) \to B_\theta^\dagger(x) = e^{i\theta_3}B^\dagger(x), \end{aligned} \tag{66}$$

for $\theta_1 = \theta_2 = 0$.

The transformations (64–66) belong to the $E(2)$ group which is the Inönü–Wigner group contraction of $SU(2)$ [18, 31]. Equations (64–66) express the dynamical rearrangement of symmetry: when the quasiparticle fields ϕ, $\phi^\dagger$, B, $B^\dagger$ undergo the $E(2)$ transformations (64–66), the $SU(2)$ transformations (2) and (9) of the Heisenberg fields ψ, $\psi^\dagger$, S^i are induced, and vice-versa. Note that (66) represents the unbroken rotation around the third axis. The derivation presented above is fully general and model independent, thus one can state that the Inönü–Wigner group contraction is the mathematical mechanism determining the rearranged symmetry group [18].

I remark that the c-number translations of the field $B(x)$ (and $B^\dagger(x)$) in (64) and (65) must be understood as the limit for $f(x) \to 1$ of the transformations

$$B(x) \to B_\theta(x) = \lim_{f(x)\to 1}\left[B(x) + if(x)\theta_1\left(\frac{M}{2}\right)^{\frac{1}{2}}\right], \tag{67}$$

$$B(x) \to B_\theta(x) = \lim_{f(x)\to 1}\left[B(x) - f(x)\theta_2\left(\frac{M}{2}\right)^{\frac{1}{2}}\right], \tag{68}$$

(and h.c.), respectively. Here the function $f(x)$ is any square-integrable function which satisfies the magnon equation. Without such function, terms like $\theta_l(\frac{M}{2\rho})K(\partial)S_\psi^{(-)}(x)$ would be contained in the quantity $A(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$ (cf. (56)), contributing, in $\mathcal{S}(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$ and $S^{(i)}(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$, to Feynman diagrams by energyless and momentumless external lines, and thus these diagrams can contain a power of zero energy singularities. In order to avoid such an infrared catastrophe one substitutes θ_i by $f(x)\theta_i$, $i = 1, 2$, and, since $B_\theta(x)$ must satisfy the magnon equation, it is necessary that $f(x)$ satisfy the magnon equation. The limit $f(x) \to 1$ must be taken at the end of the computations. Note that the magnon equations are invariant under (67) and (68) even before the limit $f(x) \to 1$, thus exhibiting the $E(2)$ invariance.

The generators of the transformations (64–66) (with θ_i, $i = 1, 2$, replaced by $f(x)\theta_i$) are

$$s_f^{(1)} = \left(\frac{M}{2}\right)^{1/2} \int d^3x [B(x) f(x) + B^\dagger(x) f^*(x)], \tag{69}$$

$$s_f^{(2)} = -i \left(\frac{M}{2}\right)^{1/2} \int d^3x [B(x) f(x) - B^\dagger(x) f^*(x)], \tag{70}$$

$$s_f^{(3)} = \int d^3x [\phi^\dagger(x) \lambda_3 \phi(x) - B^\dagger(x) B(x)]. \tag{71}$$

The introduction of the square-integrable function $f(x)$ is essential in order for the generators (69–71) to be well defined. Moreover, these generators are time independent since $f(x)$ satisfies the magnon equation. The generators (69–71) have commutation relations:

$$[s_f^{(1)}, s_f^{(2)}] = iM \int d^3x |f(x)|^2 = (\mathrm{const}) I, \tag{72}$$

$$[s_f^{(3)}, s_f^{(1)}] = i s_f^{(2)}, \qquad [s_f^{(3)}, s_f^{(2)}] = -i s_f^{(1)}, \tag{73}$$

which, in terms of $s_f^{(\pm)} = s_f^{(1)} \pm i s_f^{(2)}$, read as

$$[s_f^{(+)}, s_f^{(-)}] = 2M \int d^3x |f(x)|^2 = (\mathrm{const}) I, \tag{74}$$

$$[s_f^{(3)}, s_f^{(\pm)}] = \pm s_f^{(\pm)}. \tag{75}$$

The algebra is thus the (projective) $E(2)$ algebra and thus the generators $s_f^{(i)}$, $i = 1, 2$, (or $s_f^{(\pm)}$) exhibit their boson character (compare the above algebra (74), (75) with the Weyl–Heisenberg algebra) when expressed in terms of the quasiparticle fields ϕ and $B(x)$.

4.2 The Local $U(1)$ Gauge Model

Inspection of the dynamical maps (38–40) and of the field equations (41–44) shows that the local gauge transformations (28) of the Heisenberg fields and the transformation $B(x) \to B(x)$ are induced by the in-field transformations

$$\chi_{in}(x) \to \chi_{in}(x) + e_0 \tilde{v} Z_\chi^{-\frac{1}{2}} \lambda(x), \qquad b_{in}(x) \to b_{in}(x) + e_0 \tilde{v} Z_\chi^{-\frac{1}{2}} \lambda(x), \tag{76}$$

$$\rho_{in}(x) \to \rho_{in}(x), \qquad U_{in}^\mu(x) \to U_{in}^\mu(x). \tag{77}$$

The global transformation $\phi(x) \to e^{i\theta}\phi(x)$ is induced by

$$\chi_{in}(x) \to \chi_{in}(x) + \theta \tilde{v} Z_\chi^{-\frac{1}{2}} f(x), \tag{78}$$

$$b_{in}(x) \to b_{in}(x), \qquad \rho_{in}(x) \to \rho_{in}(x), \qquad U^{\mu}_{in}(x) \to U^{\mu}_{in}(x), \tag{79}$$

with $\partial^2 f(x) = 0$ and the limit $f(x) \to 1$ to be performed at the end of the computation. Note that under the above in-field transformations the in-field equations and the S matrix are invariant and that B is changed by an irrelevant c-number (in the limit $f \to 1$). Therefore the physical content of the theory does not change. However, under the transformations (76), Equations (29) and (31) change into

$$\langle 0|\partial_\mu A^\mu(x)|0\rangle = \partial^2 \lambda(x), \tag{80}$$

$$\langle 0|\phi(x)|0\rangle = e^{ie_0\lambda(x)}\tilde{v}, \tag{81}$$

respectively. Therefore the condition (31) is not sufficient to determine the physical content of the theory: besides $\langle 0|\phi(x)|0\rangle$, the gauge should also be specified.

In the following it will be seen that when the function $f(x)$ in the boson transformation (78) is a regular (i.e. Fourier transformable) function, its only effect is the appearance of a phase factor in the order parameter: $\tilde{v}(x) = e^{icf(x)}\tilde{v}$, with c a constant, and thus it can be eliminated by a convenient gauge transformation (gauged away). The conclusion is that when a gauge field is present, the boson transformation with regular $f(x)$ is equivalent to a gauge transformation.

The boson transformation (78) must be compatible with the Heisenberg field equations but also with the physical state condition (46): $[\chi^{(-)}_{in}(x) - b^{(-)}_{in}(x)]|a\rangle_p = 0$. Under the boson transformation (78), B changes as (cf. (43))

$$B(x) \to B(x) - \frac{e_0\tilde{v}^2}{Z_\chi} f(x), \tag{82}$$

where $\partial^2 f(x) = 0$. Equation (45) is thus violated when the physical state condition is imposed. In order to restore it, the shift in B must be compensated by means of the transformation of U_{in}:

$$U^{\mu}_{in}(x) \to U^{\mu}_{in}(x) + Z_3{}^{-\frac{1}{2}} a^\mu(x), \qquad \partial_\mu a^\mu(x) = 0, \tag{83}$$

where $a^\mu(x)$ is a convenient c-number function. The dynamical maps of the various Heisenberg operators are not affected by (83) since they contain U^{μ}_{in} and $B(x)$ in a combination such that the changes of B and of U^{μ}_{in} compensate each other provided

$$(\partial^2 + m_V^2) a_\mu(x) = \frac{m_V^2}{e_0} \partial_\mu f(x). \tag{84}$$

Equation (84) thus obtained is the Maxwell equation for the massive potential vector a_μ [40, 41]. The classical ground state current $j_{\mu,cl}$ turns out to be

$$j_{\mu,cl}(x) \equiv \langle 0|j_\mu(x)|0\rangle = m_V^2 \left[a_\mu(x) - \frac{1}{e_0}\partial_\mu f(x) \right]. \tag{85}$$

The term $m_V^2 a_\mu(x)$ is the *Meissner current*, while $\frac{m_V^2}{e_0}\partial_\mu f(x)$ is the *boson current*.

The macroscopic field and current are thus given in terms of the boson transformation function.

I remark that the classical current is related with $\partial_\mu f$, i.e. with variations in the boson transformation function.

Extension of the formalism to other gauges, such as the radiation gauge, can be found in [40].

I observe that one might also consider the possibility of *non-homogeneous* boson condensation. This is generated by translating certain asymptotic boson fields, say $\chi(x)$, (such as, but not necessarily, the NG asymptotic fields) by a space-time dependent function $f(x)$: $\chi(x) \to \chi(x) + f(x)$, where the space-time dependent functions $f(x)$ is solely constrained to be a solution of the field equation satisfied by the field $\chi(x)$. Such a transformation is called the *boson transformation*. As a result of the non-homogeneous boson condensation thus induced, a space and/or time dependent vacuum is obtained. The boson transformation theorem then holds, which states that when the boson transformation is performed on the field $\chi(x)$ in the dynamical map of the Heisenberg fields, these resulting Heisenberg fields satisfy the same field equations they had to satisfy before the boson transformation was performed [60]. In other words, the same dynamics (the same Heisenberg field equations) may describe homogeneous and non-homogeneous ground states. In particular, the function $f(x)$ may be a regular (i.e. Fourier integrable) function, or a singular function (with divergence or topological singularities). For example, $f(x)$ may be not single-valued and thus path-dependent:

$$G^{\dagger}_{\mu\nu}(x) \equiv [\partial_\mu, \partial_\nu] f(x) \neq 0, \quad \text{for certain } \mu, \nu, x. \tag{86}$$

On the other hand, $\partial_\mu f$, which is related with observables since these may be influenced by gradients in the Bose condensate, is single-valued, i.e. $[\partial_\rho, \partial_\nu]\partial_\mu f(x) = 0$. The boson condensation induced by such a path-dependent $f(x)$ may give rise to the formation of topologically non-trivial extended objects, such as vortices, monopoles, etc. It can be shown that boson condensation induced by singular functions $f(x)$, satisfying (86), may occur only in the case the field $\chi(x)$ is a massless field, as it happens in the case of NG bosons in spontaneously broken symmetry theories. In this way one can see why are extended objects with topological singularity observed only in systems showing ordered patterns [6, 59]. Moreover, only in the case of singular function $f(x)$, the boson transformation describes a transition to a unitarily inequivalent representation, and therefore to a physically distinct phase of the system (phase transition) [6, 65].

By considering the local $U(1)$ example, one sees that all the macroscopic ground state effects do not occur for regular $f(x)$ ($G^{\dagger}_{\mu\nu} = 0$). In fact, from (84) one obtains $a_\mu(x) = \frac{1}{e_0}\partial_\mu f(x)$ for regular f which implies zero classical current ($j_\mu = 0$) and zero classical field ($F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$), since the Meissner and the boson current cancel each other. The conclusion is that the vacuum current appears only when $f(x)$ has topological singularities and these can be created only by condensation of massless bosons, i.e. when spontaneous breakdown of symmetry occurs. This explains why topological defects appear in the process of phase transitions, where NG modes are present and gradients in their condensate densities are nonzero [6, 65].

On the other hand, the appearance of space-time order parameter is no guarantee that persistent ground state currents (and fields) will exist: if f is a regular function, the space-time dependence of $\tilde{v}$ can be gauged away by an appropriate gauge transformation.

Since the boson transformation with regular f does not affect observable quantities, the S-matrix (38), actually given by $\mathcal{S} =: \mathcal{S}[\rho_{in}, U^{\mu}_{in} - \frac{1}{m_V}\partial(\chi_{in} - b_{in})]:$, is in fact independent of the boson transformation with regular f:

$$\mathcal{S} \to \mathcal{S}' = :\mathcal{S}\left[\rho_{in}, U^{\mu}_{in} - \frac{1}{m_V}\partial(\chi_{in} - b_{in}) + Z_3^{-\frac{1}{2}}(a^\mu - \frac{1}{e_0}\partial^\mu f)\right]: \tag{87}$$

since $a_\mu(x) = \frac{1}{e_0}\partial_\mu f(x)$ for regular f. However, $\mathcal{S}' \neq \mathcal{S}$ for singular f. Moreover, (87) shows that $\mathcal{S}'$ includes the interaction of the quanta U_{in}^μ and ϕ_{in} with the classical field and current depending on $f(x)$ and associated to the defects. Thus quantum fluctuations are shown to interact and have effects on classically behaving macroscopic defects.

5 Group Contraction and the Infrared Effect

The equation for the field χ_{in} is invariant under the translation of the field by a constant quantity if and only if χ_{in} is a massless field, $\partial^2\chi_{in}(x,\theta) = 0$. Thus the rearrangement into the contraction of global $U(1)$ group has the same content as the Goldstone theorem [18]. Since the number operator of χ_{in} field changes under the translation, one says that coherent condensation of the boson χ_{in} occurs. The translation of the operator field by a constant is also recognized to generate coherent states.

In the following I discuss the origin of the change of the group in the process of the rearrangement of symmetry with reference to the example of the ferromagnet considered above. The conclusions are, however, general; they hold for any spontaneously broken continuous compact symmetry group of the Lagrangian. Infrared contributions to the commutations relations among the generators of the symmetry group are missed in the process of going from the Heisenberg fields to the asymptotic fields and this results in the rearrangement of the symmetry.

I decompose the magnon field into the sum of an "hard" part $B_t(x)$ and a "soft" or infrared part $B_\eta(x)$ with η infinitesimal

$$B(x) = B_t(x) + B_\eta(x), \tag{88}$$

where $B_t(x)$ contains only momenta larger than η, while momenta in $B_\eta(x)$ are smaller than η. One possible representation of $B_\eta(x)$ is

$$B_\eta(x) = \frac{1}{2}\eta\int_{-\infty}^{+\infty} dt e^{-\eta|t|}B(x) = \frac{1}{2(2\pi)^{1/2}}\,\eta\int d^3k\delta_\eta(k)B_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{89}$$

The function $\delta_\eta(k)$ approaches to $\delta(k)$ in the limit $\eta \to 0$. Therefore $B_\eta(x)$ is of order of η and independent of x in the limit $\eta \to 0$.

Now the field $B(x)$ written as in (88) is used in the expression (55) and thus the contribution of $B_\eta(x)$ to $S^{(i)}(\phi,\phi^\dagger,B,B^\dagger)$ can be obtained. By a straightforward computation whose details are given in [53], one then gets:

$$S^{(1)}(y) = s_t^{(1)}(y) + \left(\frac{1}{2M}\right)^{1/2}(B_\eta + B_\eta^\dagger)s_t^{(3)}(y), \tag{90}$$

$$S^{(2)}(y) = s_t^{(2)}(y) - i\left(\frac{1}{2M}\right)^{1/2}(B_\eta - B_\eta^\dagger)s_t^{(3)}(y), \tag{91}$$

$$S^{(3)}(y) = s_t^{(3)}(y) + \left(\frac{1}{2M}\right)^{1/2}[i(B_\eta - B_\eta^\dagger)s_t^{(2)}(y) - (B_\eta + B_\eta^\dagger)s_t^{(1)}(y)]. \tag{92}$$

These are the spin density operators expressed in terms of the quasiparticle fields and when the infrared contributions from the operators B_η and $B_\eta^\dagger$ are ignored, $s_t^{(i)}$ is obtained. Note that the matrix elements of $S^{(i)}(y)$ are equal to those of $s_t^{(i)}$:

$$\langle i|S^{(i)}(y)|j\rangle = \langle i|s_t^{(i)}(y)|j\rangle, \tag{93}$$

which shows that taking the matrix elements between physical states, i.e. smeared out (localized) states, causes the missing of the infrared contributions. In other words, the limit $\eta \to 0$ is automatically implied in the computation of the matrix elements between physical states due to the fact that physical states carry in their definition smearing out functions which act as a cutoff at infinite volume, namely, as cutoff for infrared momenta. For $i = 3$ (93) gives

$$\langle i|S^{(3)}(y)|j\rangle = \langle i|s_t^{(3)}(y)|j\rangle = M, \tag{94}$$

and therefore $s_t^{(3)}(y) = M + :s_t^{(3)}(y):$. Thus the space integration of $s_t^{(i)}(y)$ must give the generators (69–71) with B and $B^\dagger$ substituted by B_t and $B_t^\dagger$, respectively, since the cutoff $f(x)$ excludes contributions at infinite volume (the infrared contributions). By simple manipulations one obtains:

$$S_f^{(1)} = s_f^{(1)} + \left(\frac{1}{2M}\right)^{1/2} (B_\eta + B_\eta^\dagger):s_t^{(3)}:, \tag{95}$$

$$S_f^{(2)} = s_f^{(2)} - i\left(\frac{1}{2M}\right)^{1/2} (B_\eta - B_\eta^\dagger):s_t^{(3)}:, \tag{96}$$

$$S_f^{(3)} = s_t^{(3)} + \left(\frac{1}{2M}\right)^{1/2} [i(B_\eta - B_\eta^\dagger)s_t^{(2)} - (B_\eta + B_\eta^\dagger)s_t^{(1)}]. \tag{97}$$

I now observe that the spin operators $S_f^{(i)}$ satisfy the algebra for the $SU(2)$ group when the limit $f \to 1$ is taken: $[S^{(i)}, S^{(j)}] = i\epsilon_{ijk} S^{(k)}$. It is

$$[s_f^{(1)}, B_\eta^\dagger(x)] = \left(\frac{M}{2}\right)^{1/2} f_\eta(x), \tag{98}$$

and similar commutators for $s_f^{(i)}$ with $i = 2, 3$. Here $f(x) = f_t(x) + f_\eta(x)$ has been used, with same meaning for the notation as in (88). $f_\eta(x)$ contains only momenta smaller than η and thus it has a spatial domain of range $\frac{1}{\eta}$ and vanishes as $\eta \to 0$ since $f(x)$ is square-integrable. To take into account the locally infinitesimal effect, the space integration must extend to infinity. Therefore, the limit $f \to 1$ must be performed before the limit $\eta \to 0$ is taken in order to recognize the differences between $S_f^{(i)}$ and $s_f^{(i)}$. One finds

$$\begin{aligned}[S_f^{(1)}, S_f^{(2)}] = iM\int d^3x|f(x)|^2 + i\left(\frac{1}{2}\right)[f_\eta^*(x) + f_\eta(x) + f_{\bar\eta}^*(x) + f_{\bar\eta}(x)]:s_t^{(3)}: \\ - \left(\frac{1}{2M}\right)^{1/2}[i(B_{\bar\eta} - B_{\bar\eta}^\dagger)s_t^{(2)} - i(B_\eta + B_\eta^\dagger)s_t^{(1)}].\end{aligned} \tag{99}$$

Here there are two infrared cutoffs η and $\bar\eta$, and, respectively, two limits to be performed successively (no matter in which order) since two generators (i.e. two successive rotations) are involved in the commutator. Suppose $\bar\eta \ll \eta$, since $|f_{\bar\eta}(x)|^2 \ll |f_\eta(x)|^2$ one can ignore $f_{\bar\eta}$ and $f_{\bar\eta^*}$ in the r.h.s. of (99). The $f \to 1$ limit then gives

$$[S^{(1)}, S^{(2)}] = iS^{(3)}. \tag{100}$$

Moreover,

$$[S^{(3)}, S^{(1)}] = iS^{(2)}, \qquad [S^{(3)}, S^{(2)}] = -iS^{(1)}. \tag{101}$$

Therefore,

$$\lim_{\eta\to 0}\lim_{\bar{\eta}\to 0}\lim_{f\to 1}[S_f^{(i)}, S_f^{(j)}] = i\epsilon_{ijk}S^{(k)}, \tag{102}$$

while, by using $\tilde{\eta} \equiv$ (*minimum of* η *and* $\bar{\eta}$),

$$\lim_{f\to 1}\lim_{\tilde{\eta}\to 0}[S_f^{(1)}, S_f^{(2)}] = iM\lim_{f\to 1}\int d^3x|f(x)|^2 = (\text{const})I, \tag{103}$$

$$\lim_{f\to 1}\lim_{\tilde{\eta}\to 0}[S_f^{(3)}, S_f^{(1)}] = \lim_{f\to 1}\lim_{\tilde{\eta}\to 0} iS^{(2)}, \tag{104}$$

$$\lim_{f\to 1}\lim_{\tilde{\eta}\to 0}[S_f^{(3)}, S_f^{(2)}] = i\lim_{f\to 1}\lim_{\tilde{\eta}\to 0} iS^{(1)}. \tag{105}$$

Thus, if the limit $f \to 1$ is performed before the limits $\eta \to 0$ and $\bar{\eta} \to 0$, then the $SU(2)$ rotational symmetry group is obtained, while the (projective) $E(2)$ group, the group contraction of $SU(2)$, is obtained by inverting the ordering in which the limits are performed: limit $f \to 1$ and limit $\bar{\eta} \to 0$ are not commutable. The infrared term, although locally infinitesimal, gives, however, a finite global contribution to the commutators of the generators $S^{(i)}$ of the rotation group for the Heisenberg field. Its locally infinitesimal nature makes it, instead, commutable with any local operator and thus it does not contribute to the commutators of the generators for the quasiparticle fields, which are directly related to (local) observations. Therefore, the group contraction algebra is the one which is related to observable results of experiments, since the quasiparticle fields are related to observable energy levels: quasiparticles form an irreducible representation of the contraction group.

As said above, the translation of the NG modes by a constant quantity implied by the group contraction transformations are invariant transformations for the quasiparticle field equations. Therefore, also the scattering S-matrix has to be invariant under such transformations. This implies that the NG modes $B(x)$ always appear with their derivatives in the S-matrix, i.e. in the form $\partial_\mu B(x)$, and thus the NG mode interaction disappears in zero-momentum limit. In this way the so-called low-energy theorems [2, 3, 22, 70], such as the Dyson low-energy theorem for magnons in ferromagnets, the Adler theorem in high-energy physics, the soft boson limit of current algebra theory, according to which low momentum NG modes do not affect the S-matrix, are recognized to be observable manifestations of the group contraction mechanism.

Other remarkable consequences of the dynamical rearrangement of symmetry into the group contraction process are some of the relations in the current algebra formalism, such as the partially conserved axial vector currents (PCAC) and the Goldberger–Treiman relations [25, 26, 37, 60].

Summarizing, one reason why the observable symmetry group can be different from the original symmetry group is based on the fact that macroscopic observations are always a collection of local observations and therefore there always exists a possibility that in each local observation one misses an infinitesimal effect of the order of magnitude of $\frac{1}{V}$, with the volume V going to infinity. This missing effect can be accumulated as a finite amount when integrated over the whole system. Such a locally infinitesimal effect is called the infrared effect and is responsible of the origin of the difference between the algebra of the generators written in terms of the Heisenberg fields and the one of the generators written in terms of the quasiparticle or physical fields, which carry, indeed, square integrable functions (with finite spatial support). The group contraction parameter can be thus taken to be $\frac{1}{V}$ with the volume $V \to \infty$.

I stress that the result obtained above is an *exact* result, obtained without any approximation in a model independent derivation, and therefore it must not be confused with the linear approximations used sometimes in the literature, as for example when using the Holstein–Primakoff representation [30] of the $SU(2)$ group.

I also remark that in low dimensions (one or two space dimensions), the effects of the infrared NG bosons may cause the order parameter to vanish under specified conditions of temperature and dimensionality [42]. It is indeed known that there is a critical dimension D_c such that for any short-range Hamiltonian at $T \neq 0$, no spontaneous symmetry breakdown is possible. The dimension is $D_c = 1$ for discrete symmetries, while $D_c = 2$ (one space + one time dimension) for continuous symmetries. The absence of ferromagnetism in one- and two-dimensional isotropic Heisenberg model was originally observed by Mermin and Wagner [47]. Hohenberg [29] and Mermin [46] generalized this result into a statement about the non-existence of a long-range order in one- and two-dimensional systems with continuous symmetry. Analogous conclusion was reached by Coleman [16] in the framework of the quantum field theory for $(1+1)$-space-time dimensional systems in the absence of gauge fields. Infrared NG bosons are also known to destroy one-dimensional superconductivity [35].

6 Conclusions and Further Remarks

Besides the $SU(2)$ symmetry group and the $U(1)$ local gauge group considered above, other examples have been considered in the literature and, as mentioned in the Introduction, in all the cases where a (maximal) subgroup is preserved in the process of symmetry breakdown (the so-called stability group or little group of the ground state) it has been found that the group which is relevant to the observations is the group contraction of the original invariance group of the Lagrangian.

The contraction mechanism offers a powerful tool to compute the number of gapless modes occurring in the theory once the symmetry has been spontaneously broken. For example, the number of the degrees of freedom of the instanton solutions in a non-Abelian gauge theory has been computed by resorting to the mechanism of group contraction [20, 56]. Actually, spontaneous breakdown of symmetry implies that the NG fields must form an irreducible representation of the invariance group of the theory [15, 18, 24, 36, 40, 57]. The number of gapless modes is given by the number of the generators closing the Abelian subalgebra of the contracted group. These generators are linear in the fields of the NG modes and thus induce the translations of these NG bosons by constant quantities. In this way they generate boson condensation of these fields in the ground state. This is related to the appearance of macroscopic currents in the ground state which are controlled by classical equations. The mechanism of group contraction thus plays an important rôle in the passage to the macroscopic phenomena: the basic symmetry is rearranged to a contraction at observable level; in this way Abelian (boson) transformations are introduced, which regulate classical macroscopic phenomena through boson condensation. When a large number of bosons is condensed, observable symmetry patterns appear in ordered states, the quantum fluctuations become very small ($\frac{\Delta n}{n} \ll 1$) and the system behaves as a classical one. In this sense, we have *macroscopic quantum systems*. These are quantum systems not in the trivial sense that they are made, as any other physical system, by quantum components, but in the sense that their macroscopic features, such as ordering and stability, cannot be explained without recurse to the undergoing quantum dynamics. Similarly, the order parameter, which characterizes the macroscopic states of such systems, is a classical field in the sense that its measurable value (far from the critical region of phase transition) is not affected by quantum fluctuations Δn in the condensate. These results thus seem to support the conjecture [18, 51, 52, 71] that the passage from quantum to classical physics involves some group contraction phenomena.

The occurrence of group contraction in spontaneously broken symmetry theories has been also proved by means of projective geometry arguments [19] and the contraction of group representations has been proved in [13]. The contraction of group representation has been also shown [14] to provide the nonlinear realizations [12, 17, 50] of the $SU(2)$ doublet and the $SO(n)$ vector gauge theory models. As well known, nonlinear realizations provide a powerful tool of investigation in phenomenological theories where effective Lagrangians are used and spontaneous breakdown of symmetry occurs. By singling out the NG modes they make explicit the low-energy behavior of the theory and, by allowing the classification [1, 48, 49] of the allowed symmetry broken patterns, they have been used in the determination of the extrema of the most general renormalizable Higgs potentials in elementary particle physics [11].

Finally, I just mention that group contraction also plays a relevant rôle in dissipative systems and their QFT formalism [6, 9, 55] and in the dissipative quantum model of brain [66].

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